$\Leftrightarrow \forall x \in \mathbb{R}_+^*; \forall n \in \mathbb{N}^*$ on a $F_n(x) \geq \left[-\frac{1}{n}e^{-nt}\right]_0^x = \frac{1-e^{-nx}}{n}$.

• • $\forall x \in \mathbb{R}_+^*; \forall n \in \mathbb{N}^*$ on a $F_n(x) \geq \frac{1-e^{-nx}}{n}$

$\lim\limits_{x\to+\infty} \frac{1-e^{-nx}}{n} = \frac{1}{n}$ et $\lim\limits_{x\to+\infty} F_n(x)$ existe et réel

D'où $\lim\limits_{x\to+\infty} F_n(x) \geq \frac{1}{n} > 0$ par suite $\lim\limits_{x\to+\infty} F_n(x)$ est non nulle.

3/a- $u_1 = \lim\limits_{x\to+\infty} F_1(x) = \frac{\pi}{2}$ (d'après **II]1/a-**)

$u_2 = \lim\limits_{x\to+\infty} F_2(x) = 1$. (d'après **II]1/b-**)

b- $\forall n \in \mathbb{N}^*$ et $\forall t \in \mathbb{R}_+^*$ on a :

$$f^{n-1}(t)f'(t)K(t) = \left(\frac{2}{e^t+e^{-t}}\right)^{n-1} \frac{-2(e^t-e^{-t})}{(e^t+e^{-t})^2} \frac{(e^t-e^{-t})}{e^x+e^{-x}}$$

$$= \frac{2^n}{(e^t+e^{-t})^{n+2}}\left[4-(e^t+e^{-t})^2\right]$$

$$= \frac{2^{n+2}}{(e^t+e^{-t})^{n+2}} - \frac{2^n}{(e^t+e^{-t})^n} = f^{n+2}(t) - f^n(t).$$

c- $\forall n \in \mathbb{N}^*; \forall x \in \mathbb{R}_+^*$ on a :

$$\begin{pmatrix} u'(t) = f^{n-1}(t)f'(t) & \to & u(t) = \frac{1}{n}f^n(t) \\ v(t) = K(t) & \to & v'(t) = K'(t) = f^2(t) \end{pmatrix}$$

$$\int_0^x f^{n-1}(t)f'(t)K(t)dt = \left[\frac{1}{n}f^n(t)K(t)\right]_0^x - \frac{1}{n}\int_0^x f^{n+2}(t)dt$$

$$= \frac{1}{n}f^n(x)K(x) - \frac{1}{n}\int_0^x f^{n+2}(t)dt.$$

d- $\int_0^x f^{n-1}(t)f'(t)K(t)dt = \frac{1}{n}f^n(x)K(x) - \frac{1}{n}\int_0^x f^{n+2}(t)dt.$

$\Leftrightarrow \int_0^x (f^{n+2}(t) - f^n(t))dt = \frac{1}{n}f^n(x)K(x) - \frac{1}{n}\int_0^x f^{n+2}(t)dt.$

$\Leftrightarrow \int_0^x f^{n+2}(t)dt - \int_0^x f^n(t)dt = \frac{1}{n}f^n(x)K(x) - \frac{1}{n}\int_0^x f^{n+2}(t)dt.$

$\Leftrightarrow F_{n+2}(x) - F_n(x) = \frac{1}{n}f^n(x)K(x) - \frac{1}{n}F_{n+2}(x)$

$\Leftrightarrow (n+1)F_{n+2}(x) - nF_n(x) = K(x)f^n(x).$

• $\lim\limits_{x\to+\infty} [(n+1)F_{n+2}(x) - nF_n(x)] = \lim\limits_{x\to+\infty} [K(x)f^n(x)]$

$\Leftrightarrow (n+1) \lim\limits_{x\to+\infty} F_{n+2}(x) - n \lim\limits_{x\to+\infty} F_n(x) = 0$

Car $\lim\limits_{x\to+\infty} \left[\frac{e^x(1-e^{-2x})}{e^x(1+e^{-2x})} \times \left(\frac{2}{e^x+e^{-x}}\right)^n\right] = 1 \times 0 = 0$

Par suite $(n+1)u_{n+2} - nu_n = 0 \Leftrightarrow \underline{u_{n+2} = \frac{n}{n+1}u_n}$.

4/a- $n \in \mathbb{N}$.

$u_3 = \frac{1}{2}u_1$

$u_5 = \frac{3}{4}u_3$